1 System of linear equations:

Consider the matrix $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Suppose that $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ are eigenvectors of M corresponding to the eigenvalues 1 and 2 ,respectively.
In this problem, we want to find the matrix M

a

Find linear equations in the unknowns a, b, c and d

We know that $M\mathbf{x} = \lambda\mathbf{x}$

$\lambda_1 = 1$ and $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

So $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Leftrightarrow \begin{bmatrix} a+b \\ c+d \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda_2 = 2$ and $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

So $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 2 \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Leftrightarrow \begin{bmatrix} a+2b \\ c+2d \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

So then we have the equations:

$a + b = 1$

$c + d = 1$

$a + 2b = 2$

$c + 2d = 4$

b

Write down the augmented matrix corresponding to the equations found in (a)

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 0 & 2 \\ 0 & 0 & 1 & 2 & 4 \end{array}\right)$$

c

Put the augmented matrix into reduced row echelon form.

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 0 & 2 \\ 0 & 0 & 1 & 2 & 4 \end{array}\right) \rightarrow (3) = (3) - (1) \rightarrow \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 & 4 \end{array}\right) \rightarrow \text{Interchange } (2) \text{ and } (3) \rightarrow$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 & 4 \end{array}\right) \rightarrow (1) = (1) - (2) \rightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 & 4 \end{array}\right) \rightarrow (4) = (4) - (3) \rightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right)$$

$$\rightarrow (3) = (3) - (4) \rightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -4 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right)$$

d

Find all solutions a, b, c and d from the reduced row echelon form.

We can immediatly read of: $a = 0, b = 1, c = -4$ and $d = 4$

2 Determinants

Let $n \geq 1$. Suppose that $A, B \in \mathbb{R}^{n\times n}$ are diagonal matrices. Prove by induction on n that:
$\det\left(\begin{bmatrix} A & B \\ B & A \end{bmatrix}\right) = (a_{11}^2 - b_{11}^2)(a_{22}^2 - b_{22}^2)\dots(a_{nn}^2 - b_{nn}^2)$

Base case:
When $n = 1$
We see that $A \in \mathbb{R}^{1\times 1} = a_{11}$ and $B \in \mathbb{R}^{1\times 1} = b_{11}$
Therefore $\det\left(\begin{bmatrix} A & B \\ B & A \end{bmatrix}\right) = \det\left(\begin{bmatrix} a_{11} & b_{11} \\ b_{11} & a_{11} \end{bmatrix}\right) = a_{11} \cdot a_{11} - b_{11} \cdot b_{11} = a_{11}^2 - b_{11}^2$
So the statement hold for $n = 1$

Induction step:

Assume $n = k$
Assume that $\det\left(\begin{bmatrix} A_{k\times k} & B_{k\times k} \\ B_{k\times k} & A_{k\times k} \end{bmatrix}\right) = (a_{11}^2 - b_{11}^2)(a_{22}^2 - b_{22}^2)\dots(a_{kk}^2 - b_{kk}^2)$
Now we want to prove that
$\det\left(\begin{bmatrix} A_{k+1\times k+1} & B_{k+1\times k+1} \\ B_{k+1\times k+1} & A_{k+1\times k+1} \end{bmatrix}\right) = (a_{11}^2 - b_{11}^2)(a_{22}^2 - b_{22}^2)\dots(a_{kk}^2 - b_{kk}^2)(a_{(k+1)(k+1)}^2 - b_{(k+1)(k+1)})$
We know that
$\begin{vmatrix} A_{k\times k} & B_{k\times k} \\ B_{k\times k} & A_{k\times k} \end{vmatrix} = \det(A_{k\times k})\det(A_{k\times k}) - \det(B_{k\times k})\det(B_{k\times k}) = (a_{11}^2 - b_{11}^2)(a_{22}^2 - b_{22}^2)\dots(a_{kk}^2 - b_{kk}^2)$

When we calculate
$\begin{vmatrix} A_{k+1\times k+1} & B_{k+1\times k+1} \\ B_{k+1\times k+1} & A_{k+1\times k+1} \end{vmatrix} = \det(A_{k+1\times k+1}) \cdot \det(A_{k+1\times k+1}) - \det(B_{k+1\times k+1}) \cdot \det(B_{k+1\times k+1})$

$$\det(A_{k+1\times k+1}) = \begin{vmatrix} a_{11} & 0 & 0 & \dots & 0 & 0 \\ 0 & a_{22} & 0 & \dots & 0 & 0 \\ 0 & 0 & a_{33} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a_{kk} & 0 \\ 0 & 0 & 0 & \dots & 0 & a_{(k+1)(k+1)} \end{vmatrix} = a_{(k+1)(k+1)}(-1)^{(k+1)+(k+1)} \begin{vmatrix} a_{11} & 0 & 0 & \dots & 0 \\ 0 & a_{22} & 0 & \dots & 0 \\ 0 & 0 & a_{33} & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & a_{kk} \end{vmatrix}$$

$\det(A_{k+1\times k+1}) = a_{(k+1)(k+1)}(-1)^{2k+2}\det(A_{k\times k})$
By cofactor expansion along the $k + 1$th column and row.
We can conclude that $2k + 2$ is even, therefore $\det(A_{k+1\times k+1}) = a_{(k+1)(k+1)}\det(A_{k\times k})$
When we do the same for $B_{k+1\times k+1}$ we see that $\det(B_{k+1\times k+1}) = b_{(k+1)(k+1)}\det(B_{k\times k})$

$\begin{vmatrix} A_{k+1\times k+1} & B_{k+1\times k+1} \\ B_{k+1\times k+1} & A_{k+1\times k+1} \end{vmatrix} = a_{(k+1)(k+1)}\det(A_k) \cdot a_{(k+1)(k+1)}\det(A_k) - b_{(k+1)(k+1)}\det(B_k) \cdot b_{(k+1)(k+1)}\det(B_k) =$
$a_{(k+1)(k+1)}^2\det(A_k)^2 - b_{(k+1)(k+1)}^2\det(B_k)^2$

3: Vector spaces

Consider the vector space P_4. The operator $L : P_4 \to P_4$ is given by:
$L(ax^3 + bx^2 + cx + d) := (b + c)x^3 + (c + d)x^2 + (d + a)x + (a + b)$

a

Show that L is a linear operator.
$L(\alpha p + \beta q) = \alpha L(p) + \beta L(q)$

So we want to show that $L(\alpha p + \beta q) = \alpha L(p) + \beta L(q)$

Let's say that $p(x) = ix^3 + jx^2 + kx + l$ and $q(x) = ex^3 + fx^2 + gx + h$

$L(\alpha p + \beta q) = L((\alpha i + \beta e)x^3 + (\alpha j + \beta f)x^2 + (\alpha k + \beta g)x + \alpha l + \beta h)$
So then we have:
$L = \alpha(j + k)x^3 + \beta(f + g)x^3 + \alpha(k + l)x^2 + \beta(g + h)x^2 + \alpha(l + i)x + \beta(h + e)x + \alpha(i + j) + \beta(e + f)$
$L = \alpha((j+k)x^3+(k+1)x^2+(l+i)x+(i+j))+\beta((f+g)x^3+(g+h)x^2+(h+e)x+(e+f)) = \alpha L(p)+\beta L(q)$
Which is exactly what we wanted to show.

b

Find a basis for $\ker L$

$\ker(L) = \{v \in V|L(v) = 0_w\}$
$\ker(L) = \{p \in P_4|L(p) = 0_{p_4}\}$
we know that $0_{p_4} = 0x^3 + 0x^2 + 0x + 0$
So therefore we know that $b + c = c + d = d + a = a + b = 0$
And $p = ax^3 + bx^2 + cx + d$
So from that we can conclude that:

$$\begin{bmatrix}0&1&1&0\\0&0&1&1\\1&0&0&1\\1&1&0&0\end{bmatrix} \to \begin{bmatrix}1&0&0&1\\1&1&0&0\\0&1&1&0\\0&0&1&1\end{bmatrix} \to (2) = (2) - (1) \to$$

$$\begin{bmatrix}1&0&0&1\\0&1&0&-1\\0&1&1&0\\0&0&1&1\end{bmatrix} \to (3) = (3) - (2) \to \begin{bmatrix}1&0&0&1\\0&1&0&-1\\0&0&1&1\\0&0&1&1\end{bmatrix} \to (4) = (4) - (3) \to \begin{bmatrix}1&0&0&1\\0&1&0&-1\\0&0&1&1\\0&0&0&0\end{bmatrix}$$

So we see that only d is a free variable.
Therefore $a = d$ and $b = -d$ and $c = d$
So from that we can conclude that $p(x) = dx^3 - dx^2 + dx + d$
Which give us the basis $\{1, -1, 1, 1\}^T$

c

Let $S = \{p \in P_4 | p + L(p) = 0\}$
Show that S is a subspace of P_4.
Find the dimension of S

We see that $0 \in S \Rightarrow S$ is nonempty.
Let $p \in S$ and $\alpha \in \mathbb{F}$ and we know that $p + L(p) = 0$
So $\alpha p + L(\alpha p) = \alpha p + \alpha L(p)$ we have shown this in a
$\alpha p + L(\alpha p) = \alpha(p + L(p)) = \alpha \cdot 0 = 0 \in S$

Let $p, q \in S$ so $p + L(p) = 0$ and $q + L(q) = 0$
$(p + q) + L(p + q) = p + q + L(p) + L(q)$ (Shown at a) $= p + L(p) + q + L(q) = 0 + 0 = 0 \in S$
So therefore S is a subspace of P_4

Let y be an arbitrary element in S where $y = ax^3 + bx^2 + cx + d$
From that we know that $y + L(y) = 0$
So $ax^3 + bx^2 + cx + d + (b + c)x^3 + (c + d)x^2 + (d + a)x + (a + b) = (a + b + c)x^3 + (b + c + d)x^2 + (c + d + a)x + (d + a + b) = 0$
But then we can conclude $a + b + c = b + c + d = c + d + a = d + a + b = 0$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \end{array}\right) \Rightarrow \begin{array}{c} (2) = (3) \\ (3) = (4) \\ (4) = (2) \end{array} \Rightarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{array}\right) \Rightarrow \begin{array}{c} (2) = (1) - (2) \\ (3) = (1) - (3) \end{array}$$

which we can also do by first multiply it by -1, and then add row 1.

$$\Rightarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{array}\right) \Rightarrow (4) = (4) - (2) \Rightarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 2 & 0 \end{array}\right) \Rightarrow (4) = (4) - (3) \Rightarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 3 & 0 \end{array}\right)$$

$\Rightarrow (4) = \frac{1}{3}(4) \Rightarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right)$ we see that these matrix has no free variables, therefore the dimension is 4.

4

a

Find the matrix representation of D relative to the bases $E = F = (f_1, f_2, f_3, f_4, f_5, f_6)$
With V =span($(f_1, f_2, f_3, f_4, f_5, f_6)$

Theorem 4.2.1 gives us:
$\mathbf{a}_j = L(\mathbf{e}_j)$
Where $L(\mathbf{x}) = A\mathbf{x}$)
So in our situation:
$\mathbf{a}_j = D(f_j)$

We will represent $a\sin(x) + bx\sin(x) + cx^2\sin(x) + d\cos(x) + ex\cos(x) + fx^2\cos(x)$ as $(a, b, c, d, e, f)^T$

$a_1 = D(f_1) = \cos(x) = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}^T$
$a_2 = D(f_2) = \sin(x) + \cos(x) = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}^T$
$a_3 = D(f_3) = 2x\sin(x) + x^2\cos(x) = \begin{pmatrix} 0 & 2 & 0 & 0 & 0 & 1 \end{pmatrix}^T$
$a_4 = D(f_4) = -\sin(x) = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^T$
$a_5 = D(f_5) = \cos(x) - \sin(x) = \begin{pmatrix} -1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}^T$
$a_6 = D(f_6) = 2x\cos(x) - x^2\sin(x) = \begin{pmatrix} 0 & 0 & -1 & 0 & 2 & 0 \end{pmatrix}^T$

So the matrix representation is given by the following matrix:

$$A = \begin{bmatrix} 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

b

Use the matrix representation found in (a) to find the definite integral.

$\int ax\cos(x) + bx\sin(x)dx$
So $D(f) = f' = ax\cos(x) + bx\sin(x)dx$
What is f?

$$D(f) = Af = \begin{bmatrix} 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} f = \begin{bmatrix} 0 \\ b \\ 0 \\ 0 \\ a \\ 0 \end{bmatrix} \text{ which give us:}$$

$$\left(\begin{array}{cccccc|c} 0 & 1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & a \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \left(\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 2 & a \\ 0 & 0 & 0 & 0 & 1 & 0 & -b \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

I did it on paper, but it was not readable and not enough time to type it.

So we have $x_6 = -1$ and $x_5 = -b$ and $x_4 + 2x_6 = a \Rightarrow x_4 = a$,and $x_3 = 0$ and $x_2 - x_4 = 0 \Rightarrow x_2 - a = 0 \Rightarrow x_2 = a$ and finally $x_1 + x_5 = 0 \Rightarrow x_1 - b = 0 \Rightarrow x_1 = b$
$f = bx\sin(x) + ax^2\cos(x) + ax\cos(x) - b\sin(x)$

Question 5:

Let $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\mathbf{y}_1 = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ and $\mathbf{y}_2 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$
Be given vectors. We want to find the best least squares fit for $\mathbf{y} = M\mathbf{x}$ where M is a symmetric matrix of the form $M = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$

a

Find the normal equations:

By $\mathbf{y} = M\mathbf{x}$ we find:
$4 = a + b$,$3 = a + b$,$2 = a + b$ and $2 = b + a$ so we have:
$\left(\begin{array}{cc|c} 1 & 1 & 4 \\ 1 & 1 & 3 \\ 1 & 2 & 2 \\ 2 & 1 & 2 \end{array}\right)$ $A^T A\mathbf{x} = A^T\mathbf{b}$ So:

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 1 & 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 1 & 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 2 \\ 2 \end{bmatrix}$$

b

Find the solution.
$A^T A = \begin{bmatrix} 7 & 5 \\ 5 & 7 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 13 \\ 13 \end{bmatrix}$ $\Rightarrow$ same reasoning as before $\Rightarrow$ $\left(\begin{array}{cc|c} 1 & \frac{8}{7} & \frac{13}{7} \\ 0 & 1 & \frac{13}{15} \end{array}\right)$ so $b = \frac{13}{15}$ and $a = \frac{13}{15}$
So $M = \begin{bmatrix} \frac{13}{15} & \frac{13}{15} \\ \frac{13}{15} & \frac{13}{15} \end{bmatrix}$